Classical dynamics of a charged particle in a laser field beyond the dipole approximation

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Abstract

The classical dynamics of a charged particle travelling in a laser field modelled by an elliptically polarized monochromatic electromagnetic plane wave is discussed within the time reparametrization invariant form of the non-relativistic Hamilton-Jacobi theory. The exact parametric representation for a particle's orbit in an arbitrary plane wave background beyond the dipole approximation and including effect of the magnetic field is derived. For an elliptically polarized monochromatic plane wave the particle's trajectory, as an explicit function of the laboratory frame's time, is given in terms of the Jacobian elliptic functions, whose modulus is proportional to the laser's intensity and depends on the polarization of radiation. It is shown that the system exposes the *intensity duality*, correspondence between the motion in the backgrounds with various intensities. In virtue of the modular properties of the Jacobian functions, by starting with the representative "fundamental solution " and applying a certain modular transformations one can obtain the particle's orbit in the monochromatic plane wave background with arbitrarily prescribed characteristics.

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Contents

I. INTRODUCTION

By the end of the 20th century Quantum Electrodynamics set an undoubted example of a theory allowing a thorough understanding and the ability to make predictions to an arbitrary precision the interaction of individual fundamental particles, photons and electrons. Until the development of the laser existing photon sources produced incoherent photons one at a time and the perturbative QED, based on an elementary *single photon* interacting with a *single electron*, was a perfectly adequate and accurate theory despite not taking into account the *coherence* aspect of the photons. Since the 1960's, after the advances in laser technology, the first theoretical studies of collective multi-photon relativistic dynamics began. Over the past fifteen years the revolutionary invention for a new technique of laser pulse amplification has led to the possibility to create the attosecond pulses of light beams of a very high intensity and made the practical study of a new area of QED, *physics of high-intensity laser-matter interactions* possible (see e.g., textbooks [1] and recent surveys [2],[3]). Contemporary renaissance of these research activities has opened up many new promising opportunities to develop our understanding of fundamental physics as well as the possibility of absolutely new applications.

The necessity to modify the conventional single particle description of radiation-matter interaction at high intensities is arguably best illustrated by classical Thomson scattering, i.e. scattering of an electromagnetic wave by a free electron. The theory of ordinary Thomson scattering [4, 5] assumes that a linearly polarized monochromatic plane wave's phase [1] is approximated as $\omega_L t - \boldsymbol{k}_L \cdot \boldsymbol{x} \approx \omega_L t$ (*dipole approximation*) and the particle at rest responds only to the electric field

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos \omega_L t\,.$$

Therefore the electron consequently executes simple harmonic motion at the same frequency as the wave, ω_L, along the electric field direction:

$$\boldsymbol{x} = -\frac{e}{m\omega_L^2}\,\boldsymbol{E}_0\,\cos \omega_L t\,. \tag{1}$$

The radiation from such an oscillating charge is calculated with the aid of the classical

[1] The monochromatic electromagnetic plane wave represents the simplest way to mathematically model a laser field, which is a reasonable assumption providing the transverse directions of the laser beams are much larger than the dimensions of the system considered.

Larmor radiation formula [5] which results in the universal, frequency independent, total cross section

$$\sigma_{\mathrm{T}} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 . \tag{2}$$

The assumption that the magnetic part of the Heaviside-Lorentz force of the plane wave,

$$\boldsymbol{F} = e\,\boldsymbol{E} + \frac{e}{c}\,\boldsymbol{v}\,\times\,\boldsymbol{B}\,,$$

can be ignored,

$$\frac{v}{c}\,||\boldsymbol{B}_0|| \ll ||\boldsymbol{E}_0||\,,$$

implies, taking into account the equality, $||\boldsymbol{E}_0|| = ||\boldsymbol{B}_0||$, that the electron motion should be completely non-relativistic, $v \ll c$. According to (1) the maximum electron velocity is $v_{max} = e||\mathbf{E}_0||/m\omega_L$ and therefore the non-relativistic character of motion will hold true providing

$$\eta^2 := \frac{e^2}{\omega_L^2 m^2 c^2}\,\boldsymbol{E}_0^2 = \frac{2}{\pi}\,\frac{e^2\lambda_L^2}{m^2c^5}\,I_L \ll 1\,. \tag{3}$$

Here λ_L is the radiation wavelength and the beam intensity $I_L := c\,\boldsymbol{E}_0^2/8\pi$ has been introduced.

When the dimensionless *intensity parameter* η is sufficiently large the dipole approximation is no longer valid and magnetic force cannot justifiably be ignored. The electron motion becomes a nonlinear function of the driving force and relativistic effects will modify photon-electron scattering. The classical fully relativistic four-dimensional treatment of the light-electron scattering when the electromagnetic radiation is modelled by an electromagnetic monochromatic plane is well established. In this background the classical relativistic equation of motion for the electron with the full Heaviside-Lorentz force and taking into account the retardation effect can be solved exactly. This provides with an implicit parametric representation for the electron trajectory[2]. In this solution the relativistic particle's position vector $\boldsymbol{x}$ is given as a function of the proper time which is in turn a function of $\boldsymbol{x}$. The explicit solution $\boldsymbol{x}(t)$ is a subtle function of the physical time, known only in the form of an infinite series expansion over the harmonics with a fundamental frequency depending

[2] To the best of our knowledge J.Frenkel in 1925 was the fist one who presented the parametric relativistic solution [6]. Since that publication this problem has been studied many times in different context (for the principal references see reviews [2],[3] and original papers [7]) and became at issue for classical textbooks [8, 9, 10]. The brief history of strong-field physics written by the pioneer of this area can be found in the recent book [11].

on the radiation intensity [12]. The appearance of the higher harmonics in the scattered radiation is a direct manifestation of this fact and results in the modification of the classical Thomson cross section (2) for a high-intensity laser beam (see, for example, the basic references [12, 13] and for the recent direct experimental observation of the second harmonics [14]).

In order to fully understand the relativistic effects it is essential to have a clear understanding of how the transition from non-relativistic motion to the relativistic motion occurs. Analysis (see for details [15, 16]) of the deviations from the non-relativistic regime shows the possibility to categorize the relativistic corrections into two groups; the first one is due to the influence of the magnetic part, $\boldsymbol{v} \times \boldsymbol{B}$, of the Heaviside-Lorentz force and gives the first-order relativistic effects $O(v/c)$. The second category are the "true relativistic" second-order effects $O(v^2/c^2)$ such as, for example, the well-known relativistic corrections to the kinetic energy, Darwin term and spin-orbit coupling. It was shown by H.Reiss [15, 16], that there is an intermediate regime with a wide range of parameters (in terms of the laser's intensity & frequency) where the leading contribution to the deviation from the non-relativistic description comes from the magnetic fields effects, while the "truly relativistic" v^2/c^2 effects can be still be neglected. This important observation gives justifies exploiting the combined methods, where the relativistic effects are taken into account only partially as corrections. Such an approach is, in particular, very attractive to the description of laser-atom interactions, where the whole formalism is intrinsically non-relativistic (see the recent publication [17] and references therein). It is also important since the widely used numerical technique becomes very cumbersome when passing to the fully four dimensional relativistic covariant description.

Several partially relativistic approaches, taking into account magnetic field's influence on the particle's dynamics were developed (see, for example,[15, 16, 18] and references therein). The straightforward way to take into account the magnetic field is to go beyond the dipole approximation by including the full phase dependence $\omega_L t - \boldsymbol{k}_L \cdot \boldsymbol{x}$ in the corresponding plane wave potential[3]. Since the exact parametric solution to the relativistic Lorentz equation for a charged particle in plane wave background is known one could naively expect that the cor-

[3] Note however, the discussions in [19, 20] of the possible artifacts arising from this way of partial relativistic consideration as well as necessity to include the radiation damping effects [21, 22].

responding Newton equation can be solved exactly in an even more simple form. However, surprisingly, we were not able to find (in the extensive literature on laser-matter interactions) such an exact solution to Newton's equations of motion for a charged particle in plane wave background. In this note we aimed to fill this gap by providing the description of the intermediate region of the laser-particle interaction and to present an exact parametric solution to the non-relativistic equations of motion that is in a close analogy to the corresponding relativistic problem. As an application of the general formulae we will consider in detail the case of an elliptically polarized monochromatic plane wave. For this case we derive the explicit representation of a charged particle's orbit in terms of the laboratory frame's time. The solution is given in terms of the Jacobian elliptic functions, whose modulus depends on the background's radiation intensity and polarization. The particle's motion represents a drift displacement and an infinite sum of harmonic oscillations with the fundamental oscillation frequency depending on the radiation intensity in a nonlinear way. Furthermore, owing to the modular properties of the Jacobian functions which is the existence of relations between elliptic functions with different periods, we attest an interesting *duality* between motions in backgrounds with a various intensities regimes. In particular, we show how a particle's trajectory in a monochromatic plane wave with a high intensity can be obtained from the *"fundamental solution,"* which describes the motion in a background with low intensity.

The presentation of the material in the article is gathered as follows. In Section II the non-relativistic motion of a charged particle in an external electromagnetic background is reformulated in a time reparametrization invariant fashion. The equal footing of time and space coordinates mimics the relativistic theory and enables us to use the conventional stationary Hamilton-Jacobi method to find a parametric solution to the non-relativistic equations of motion for a charged particle travelling in an arbitrary plane wave background. In Sections III and IV we discuss in details a particle trajectories in the monochromatic plane wave background with an arbitrary elliptic polarization. In Appendix A we briefly comment on the instantaneous implementation of a gauge symmetry and Galilean boost transformation for a classical "non-relativistic" particle in an electromagnetic background. Appendix B sketches the derivation of a particle's trajectory for generic boundary conditions. Finally, in order to make article self-contained, Appendix C gives some mathematical features of the Jacobian elliptic functions used throughout the main text.

II. NON-RELATIVISTIC PARTICLE IN AN EXTERNAL FIELD

A point non-relativistic particle, with mass m and electric charge $-e$, moving in an external electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ is influenced by the Heaviside-Lorentz force. A particle's trajectory $\boldsymbol{x}(t)$ may be determined from Newton's equations of motion

$$m\,\frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2}\; = e\,\boldsymbol{E}(t,\boldsymbol{x}(t)) + \frac{e}{c}\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\;\times \boldsymbol{B}(t,\boldsymbol{x}(t))\,. \tag{4}$$

The nonlinear equations (4) can be reproduced within the conventional variational principle of least action based on the following "non-relativistic" Lagrangian function[4]

$$\mathcal{L}\left(\boldsymbol{x}\,,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\,,t\right) = \frac{m}{2}\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \frac{e}{c}\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot\;\boldsymbol{A}(t,\boldsymbol{x}(t)) - e\,\Phi(t,\boldsymbol{x}(t)) \tag{5}$$

if the external electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ are defined in terms of the gauge potential $A^\mu(t,\boldsymbol{x}) = (\Phi(t,\boldsymbol{x})\,,\boldsymbol{A}(t,\boldsymbol{x}))$ in the standard way

$$\boldsymbol{E}(t,\boldsymbol{x}) \;=\; \nabla\Phi(t,\boldsymbol{x}) - \frac{1}{c}\frac{\partial}{\partial t}\,\boldsymbol{A}(t,\boldsymbol{x})\,, \tag{6}$$

$$\boldsymbol{B}(t,\boldsymbol{x}) \;=\; \nabla\times\boldsymbol{A}(t,\boldsymbol{x})\,. \tag{7}$$

Here we intend to solve (4) for a special case of idealized laser field described by the so-called electromagnetic *monochromatic plane wave*. However, before restricting ourselves to this special case, we consider at first the more general plane wave background [8] with a gauge potential of the form

$$A_\mu(\boldsymbol{x},t) = A_\mu(\xi)\,, \tag{8}$$

where A_μ is a 4-vector depending only on the light-cone coordinate

$$\xi = t - \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c}\,, \tag{9}$$

with a unit constant 3-vector $\boldsymbol{n}$ pointing in the direction of a wave propagation. Also we assume that the Coulomb gauge is imposed which reduces to the condition

$$\boldsymbol{n}\cdot\boldsymbol{A} = 0\,. \tag{10}$$

[4] The name "non-relativistic" is somewhat misleading because the Lagrangian (5) is not Galilean invariant, one can speak only about an approximate Galilean symmetry for small particle velocities (see discussion in [23] and Appendix A).

Bellow it will be shown that the Lagrangian system (5) with the plane wave background of type (8) is classically integrable and its solution can be represented in a parametric integral form. Furthermore, after specialising to the monochromatic plane wave background with an arbitrary elliptic polarization, we derive the explicit form of the trajectory in terms of the well-known Jacobian elliptic functions. To demonstrate this we exploit the ideas from classical Hamilton-Jacobi theory [24, 25, 26] as well as the Dirac constraint formalism [27].

A. The Dirac parametrization trick and the Hamilton-Jacobi equation

Due to the explicit time dependence of the electromagnetic wave potential the Lagrangian (5) describes a *non-autonomous system*. It is convenient to enlarge the corresponding configuration space in such a way that the resulting extended system becomes autonomous at the expense of being invariant under an arbitrary time parametrization. This method is often used in classical mechanics (see, for example, [25], page 90 or [26], page 235) and also known to particle physicists as the Dirac "parametrization trick" [27, 28].

The basic elements of this approach are the following. Starting from an arbitrary Lagrangian system with Lagrangian $\mathcal{L}\big(\boldsymbol{x}(t)\,, \mathrm{d}\boldsymbol{x}/\mathrm{d}t\,, t\big)$ the configuration space is extended by considering time t as a new dynamical variable $t(s)$, which, together with the other "spatial" coordinates $\boldsymbol{x}(s)$, depends upon the auxiliary evolution parameter s. The dynamics of the extended system is determined from the degenerate, homogeneous (degree one), time-reparametrization invariant Lagrangian $\mathcal{L}^*$ constructed from the initial $\mathcal{L}$ according to the rule:

$$\mathcal{L}^*\left(\boldsymbol{x}(s)\,, t(s)\,, \dot{\boldsymbol{x}}(s)\,, \dot{t}(s)\right) := \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)\mathcal{L}\left(\boldsymbol{x}(s)\,, \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}\Big/\frac{\mathrm{d}t}{\mathrm{d}s}\,,\ t(s)\right). \tag{11}$$

Hereafter a "dot" over a variable denotes a derivative with respect to the evolution parameter s and we require that $t(s)$ is a monotonic, increasing function of the new evolution parameter s

$$\frac{\mathrm{d}t}{\mathrm{d}s} > 0\,. \tag{12}$$

Numerically the new classical action based on the extended Lagrangian (11) is the same as the action of the initial system, however, it turns out to be invariant with respect to an arbitrary monotonic change of the evolution parameter

$$s \to s' = f(s)\,. \tag{13}$$

The extended phase of the system (11) consists of $3+1$ canonical pairs

$$\boldsymbol{Z}(s) := \begin{bmatrix} \boldsymbol{x}(s)\,, & \boldsymbol{p}(s) \\ t(s)\,, & p_t(s) \end{bmatrix}\,, \qquad \boldsymbol{p} := \frac{\partial \mathcal{L}^*}{\partial \dot{\boldsymbol{x}}(s)}\,, \qquad p_t := \frac{\partial \mathcal{L}^*}{\partial \dot{t}(s)}\,, \tag{14}$$

but due to the parametrization invariance (13) the dynamics are constrained to develop on the surface of the phase space defined by the equation

$$\mathcal{H} := p_t + H_c = 0\,, \tag{15}$$

where H_c is the canonical Hamiltonian corresponding to the initial Lagrangian $\mathcal{L}$.

According to the Hamilton-Dirac description [27, 28] the constraint (15) plays a twofold role. First, it is a generator of the local symmetry transformation of the phase space coordinates (14) induced by the time reparametrization (13). Second, inasmuch as the canonical Hamiltonian derived from the Lagrangian $\mathcal{L}^*$ is identically zero the dynamics are encoded in the constraint (15) also. This constraint generates the evolution of the extended system via the Hamilton-Dirac equations

$$\dot{\boldsymbol{Z}} = \lambda(s)\{\boldsymbol{Z}\,, \mathcal{H}\}\,, \tag{16}$$

with an arbitrary function $\lambda(s)$. The arbitrariness of the Lagrange multiplier $\lambda(s)$ reflects the freedom to use an arbitrary evolution parameter. Fixing it, by imposing an additional constraint

$$\chi(s, \boldsymbol{x}, t) = 0\,,$$

allows one to show that any solution to (16) either coincides with the classical trajectory of the initial Lagrangian $\mathcal{L}$, if gauge $\chi := t - s$ is chosen, or by using any other admissible gauge, is canonically equivalent to it [28].

Applying this general scheme to (5) the Lagrangian is transformed to

$$\mathcal{L}^*\left(\boldsymbol{x}\,, \dot{\boldsymbol{x}}\,, t\,, \dot{t}\right) = \frac{m}{2}\left(\frac{\dot{\boldsymbol{x}}}{\dot{t}}\right)^2 \dot{t} + \frac{e}{c}\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A}(\xi) - e\,\dot{t}\,\Phi(\xi) \tag{17}$$

and the time reparametrization invariant Hamiltonian dynamics of the non-relativistic particle is governed by the following Hamiltonian constraint

$$\mathcal{H} := p_t + e\Phi + \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\,\boldsymbol{A}\right)^2 \;=\; 0\,. \tag{18}$$

Our plan to find the trajectory of a non-relativistic particle in an electromagnetic plane background is the following. At first, exploiting the Hamilton-Jacobi method, we perform a canonical transformation to a free theory. Having the explicit form of this canonical transformation as well as the solution to the free equations of motion we will be able to write down the solution to the Hamilton-Dirac equation of motion (16). Then, fixing with a suitable gauge, the Lagrange multiplier function $\lambda(s)$ will be determined and finally the trajectories of a charged particle will be given in a parametric form analogous to the corresponding relativistic problem.

B. Canonical transformation to a free system

Following the basic idea of the Hamilton-Jacobi method consider a canonical transformation

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{x}(s)\,, & \boldsymbol{p}(s) \\ t(s)\,, & p_t(s) \end{bmatrix} \qquad \longleftrightarrow \qquad \boldsymbol{Z}_0 = \begin{bmatrix} \boldsymbol{X}(s)\,, & \boldsymbol{\Pi}(s) \\ T(s)\,, & \Pi_T(s) \end{bmatrix}\,, \tag{19}$$

that turns the constraint (18) into the constraint of a free theory

$$\mathcal{H}_0 = \Pi_T + \frac{1}{2m}\boldsymbol{\Pi}^2 = 0\,. \tag{20}$$

This canonical transformation "absorbs " the electromagnetic field and as a result the constraint $\mathcal{H}_0$ generates according to (16) the simple free evolution:

$$T(s) = T_0 + \int_0^s \mathrm{d}u\,\lambda(u)\,, \qquad \boldsymbol{X}(s) = \boldsymbol{X}_0 + \frac{\boldsymbol{\Pi}}{m}\int_0^s \mathrm{d}u\,\lambda(u)\,. \tag{21}$$

In this solution Π_T and $\boldsymbol{\Pi}$ denote constants of motion which contain all the information about the initial position and velocity of the particle when $s = 0$. The knowledge of the explicit form of the "absorbing transformation" $\boldsymbol{Z} \to \boldsymbol{Z}_0$ is equivalent to solving the initial interacting problem and can be found in exceptional cases only. Fortuitously, it is the case for the problem we are considering here.

The "absorbing" transformation (19) can be established within the well-known method of generating function [24, 25] using the S_2-function of old coordinates $(t, \boldsymbol{x})$ and new momenta $(\Pi_T, \boldsymbol{\Pi})$ of the form

$$S_2(t, \boldsymbol{x}, \Pi_T, \boldsymbol{\Pi}) = t\,\Pi_T + \boldsymbol{x}\cdot\boldsymbol{\Pi} + \mathcal{F}(\xi, \boldsymbol{\Pi})\,, \tag{22}$$

with the unknown function $\mathcal{F}(\xi, \mathbf{\Pi})$ to be determined as follows. Write the old momenta p_t and $\boldsymbol{p}$ as function of transformed coordinates

$$p_t \;=\; \frac{\partial S_2}{\partial t} = \Pi_T + \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\xi}\,, \tag{23}$$

$$\boldsymbol{p} \;=\; \frac{\partial S_2}{\partial \boldsymbol{x}} = \mathbf{\Pi} - \frac{\boldsymbol{n}}{c}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\xi}\,. \tag{24}$$

Decompose all 3-vectors into their orthogonal and parallel components with respect to the direction of wave propagation, e.g., $\mathbf{\Pi} = \mathbf{\Pi}_\perp + \Pi_\| \,\boldsymbol{n}$, and by using the gauge condition (10) we see that the constraint (18) reduces to a free constraint (20) if the function $\mathcal{F}$ is a solution to the equation

$$\left[\frac{1}{c}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\xi} + (mc - \Pi_\|)\right]^2 = (mc - \Pi_\|)^2 + W(\xi, \mathbf{\Pi}_\perp)\,, \tag{25}$$

where

$$W(\xi, \mathbf{\Pi}_\perp) := -\frac{e^2}{c^2}\,\boldsymbol{A}_\perp^2 + 2\,\frac{e}{c}\,\boldsymbol{A}_\perp\cdot\mathbf{\Pi}_\perp + 2me\,\Phi\,. \tag{26}$$

The left hand side of (25) is positive definite, therefore, a solution to (25) is real function if

$$(mc - \Pi_\|)^2 + W(\xi, \mathbf{\Pi}_\perp) \geq 0\,. \tag{27}$$

When the condition (27) is satisfied and imposing the boundary condition $\mathcal{F}(0) = 0$, we have the solution:

$$\mathcal{F}(\xi\,, \mathbf{\Pi}) = -c(mc - \Pi_\|)\,\xi + c\int_0^\xi \mathrm{d}u\,\sqrt{(mc - \Pi_\|)^2 + W(u, \mathbf{\Pi}_\perp)}\,. \tag{28}$$

Here we insist that the inequality (27) is satisfied identically for all values of integration variable u. The importance of the restriction (27) will be discussed later by analyzing the special case of a monochromatic wave background.

Now substituting the function $\mathcal{F}$ from (28) into (23) and (24) one can easily determine the expressions for a new momenta as function of the initial ones. The momentum which is canonically conjugated to the new time coordinate T is

$$\Pi_T \;=\; p_t - c\left[(mc - p_\|) + \sqrt{(mc - p_\|)^2 - W(\xi\,, \boldsymbol{p}_\perp)}\right], \tag{29}$$

and the new three dimensional momenta read

$$\mathbf{\Pi}_\perp = \boldsymbol{p}_\perp\,, \quad \Pi_\| = mc + \sqrt{(mc - p_\|)^2 - W(\xi\,, \boldsymbol{p}_\perp)}\,. \tag{30}$$

Using the generating equations

$$T = \frac{\partial S_2}{\partial \Pi_T} = t\,, \tag{31}$$

$$\boldsymbol{X} = \frac{\partial S_2}{\partial \boldsymbol{\Pi}} = \boldsymbol{x} + \left.\frac{\partial \mathcal{F}}{\partial \boldsymbol{\Pi}}\right|_{\boldsymbol{\Pi}=\boldsymbol{\Pi}(t,\boldsymbol{x},\boldsymbol{p})}, \tag{32}$$

one can find the new coordinates as a function of the old ones. According to these results the time coordinate is unchanged

$$T = t\,, \tag{33}$$

while the new three dimensional coordinates are

$$\boldsymbol{X}_\perp = \boldsymbol{x}_\perp + \frac{e}{|mc - p_\parallel|}\int_0^\xi \mathrm{d}u\; \boldsymbol{A}_\perp(u)\,, \tag{34}$$

$$X_\parallel = x_\parallel + c\xi + \frac{c}{|mc - p_\parallel|}\int_0^\xi \mathrm{d}u\; \sqrt{\left(mc - p_\parallel\right)^2 - W(u, \boldsymbol{p}_\perp)}\,. \tag{35}$$

Equations (30) express three independent constants of motion, the two first constants coincide with the transversal momenta $\boldsymbol{p}_\perp$, while the third one $\Pi_\parallel$, can be interpreted as the longitudinal momenta of particle only in the asymptotic region where the interaction with the electromagnetic field is negligible[5]. Note also, from (29) and (30), it follows that the so-called *"light-cone energy"* represents a constant of motion

$$\frac{p_t}{c} + p_\parallel = \frac{\Pi_T}{c} + \Pi_\parallel = \textit{ constant}\,. \tag{36}$$

Suppose now that we are able to invert equations (34)-(35), i.e. express the old canonical pairs $(t\,, p_t)$ and $(\boldsymbol{x}\,, \boldsymbol{p})$ as functions of $(T\,, \Pi_t)$ and $(\boldsymbol{X}\,, \boldsymbol{\Pi})$. Further, since $(T\,, \Pi_t)$ and $(\boldsymbol{X}\,, \boldsymbol{\Pi})$ are known from (21) (up to a gauge fixing), these inverted expressions give the solution to the Hamilton-Dirac equations (16). To succeed in this inversion it is necessary to impose an appropriate gauge condition on the coordinates t and $\boldsymbol{x}$.

C. Light-cone gauge fixing and parametric solution

The observation that the light cone energy (36) is a constant of motion suggests a natural gauge fixing condition. Namely, the evolution parameter s can be identified with the

[5] For further interpretation see equations (54) and (55) below.

canonical variable conjugated to the light cone energy

$$\chi := t(s) - \frac{x_{\|}(s)}{c} - s = 0\,. \tag{37}$$

To find the Lagrange multiplier function λ, note that according to the solution (21)

$$T - \frac{X_{\|}}{c} = (1 - \frac{\Pi_{\|}}{mc}) \int_0^s du\, \lambda(u)\,, \tag{38}$$

where the initial conditions T_0 and $\boldsymbol{X}_0$ have been set equal to zero for simplicity. Furthermore, by using this relation and the equations (33)-(35) with the gauge fixing condition (37) implemented, we find that the Lagrange multiplier obeys the following integral relation

$$\int_0^s du\, \lambda(u) = \frac{mc}{\omega} \int_0^{\omega s} du\, \frac{1}{\sqrt{(\Pi_{\|} - mc)^2 + W(u\,, \mathbf{\Pi}_\perp)}}\,. \tag{39}$$

With the aid of (39) the equations (33)-(35) can be inverted with respect to the initial coordinates

$$t(s) \;=\; mc \int_0^s \mathrm{d}u\, \frac{1}{\sqrt{(\Pi_{\|} - mc)^2 + W(u, \mathbf{\Pi}_\perp)}}\,, \tag{40}$$

$$x_{\|}(s) \;=\; -cs + mc^2 \int_0^s \mathrm{d}u\, \frac{1}{\sqrt{(\Pi_{\|} - mc)^2 + W(u, \mathbf{\Pi}_\perp)}}\,, \tag{41}$$

$$\boldsymbol{x}_\perp(s) \;=\; c \int_0^s \mathrm{d}u\, \frac{\mathbf{\Pi}_\perp - \frac{e}{c}\mathbf{A}_\perp(u)}{\sqrt{(\Pi_{\|} - mc)^2 + W(u, \mathbf{\Pi}_\perp)}}\,. \tag{42}$$

The formulae (40)-(42) gives the parametric solution for a non-relativistic particle's trajectory in an arbitrary plane wave background. They are in a close analogy with the parametric solution of the corresponding relativistic problem [8, 9, 10].

In order to write down the three dimensional trajectory $\boldsymbol{x}(t)$, as a function of the physical time, it is necessary to invert (40), i.e., to find s as function of t. It is not possible to write down an explicit formula for an arbitrary plane wave background. However, in the Section III, it will be shown how to solve this problem for a monochromatic plane wave.

D. A particle orbit in a weak plane wave background

Before considering the inversion of the integral (40) for a monochromatic plane wave we will sketch the possibility to solve the same problem for an arbitrarily "weak" plane wave in the form of and expansion in the intensity parameter.

According to (40)-(42), if s as a function of t is known, $s = f^{-1}(t)$, the classical trajectory can written in the form of the integral

$$\boldsymbol{x}_\perp(t) = \frac{\mathbf{\Pi}_\perp}{m}\, t - \frac{e}{mc} \int_0^t \mathrm{d}t'\, \boldsymbol{A}_\perp(f^{-1}(t'))\,, \tag{43}$$

and

$$z(t) = ct - cf^{-1}(t)\,, \tag{44}$$

These formulae, with the function f^{-1} determined from (40), gives a non-relativistic particle's trajectory as function of LAB frame time in an arbitrary plane wave background.

A "naive" non-relativistic limit of the solution for a particle's trajectory follows from (43) and (44) by assuming the validity of the formal $1/c$ expansion of the denominator of the integrand in the expression (40). In this case, for small enough laser intensities, a closed form of the charged particle's classical trajectory, $\boldsymbol{x}(t)$, as function of the LAB frame time t can be written straightforwardly.

Indeed, in the approximation $mc - \Pi_{\|} \approx mc$ and $\mathbf{\Pi}_\perp/mc \approx 0$, keeping only the leading term of the $1/mc$-expansion of the denominator in (40) we have

$$t(s) = s + \frac{1}{2}\eta^2 \int_0^s \mathrm{d}u\, \boldsymbol{a}_\perp^2(u)\,. \tag{45}$$

In (45) the normalized potential $\boldsymbol{a}_\perp := \boldsymbol{A}_\perp/\sqrt{\langle \boldsymbol{A}_\perp^2 \rangle}$ with $\langle \dots \rangle$ denoting the time average and the dimensionless intensity parameter η,

$$\eta^2 = -2\, \frac{e^2}{m^2c^4}\, \langle A_\mu\, A^\mu \rangle\,, \tag{46}$$

have been introduced.

Therefore, for small intensities, the auxiliary time s in the leading η order is

$$s = t - \frac{1}{2}\eta^2 \int_0^t \mathrm{d}u\, \boldsymbol{a}_\perp^2(u) + \dots\,, \tag{47}$$

and the approximate form of a charged particle's trajectory reads

$$\boldsymbol{x}_\perp(t) = \frac{\mathbf{\Pi}_\perp}{m}\, t - c\eta \int_0^t \mathrm{d}u\, \boldsymbol{a}_\perp(u) + \frac{\mathbf{\Pi}_\perp}{2m}\,\eta^2 \int_0^t \mathrm{d}u\, \boldsymbol{a}_\perp^2(u) + \dots\,, \tag{48}$$

$$z(t) = \frac{1}{2}\, c\eta^2 \int_0^t \mathrm{d}u\, \boldsymbol{a}_\perp^2(u) + \dots\,. \tag{49}$$

The higher order corrections can be obtained in a similar way using, for example, the well-known Lagrange expansion method over the small parameter [29].

III. PARTICLE'S ORBIT IN A MONOCHROMATIC PLANE WAVE

In this section we exploit the generic parametric representation for a particle's trajectory found above for the practically important case of a charge's propagation in the background of a monochromatic plane wave with an arbitrary polarization.

We specify the gauge potential in equations (40) - (42) as

$$A^\mu := a(u)\left(0\,,\ \varepsilon\cos(u)\,,\ \sqrt{1-\varepsilon^2}\sin(u)\,,\ 0\right), \qquad u = \omega_L\left(t - \frac{z}{c}\right)\,. \tag{50}$$

Here, the four vector in brackets describes a monochromatic elliptically polarized electromagnetic plane wave with frequency ω_L travelling in the z-direction. The parameter $0 \leq \varepsilon \leq 1$ measures the polarization in a way such that the boundary values $\varepsilon = 0$ and $\varepsilon = 1$ correspond to linear polarization, while $\varepsilon = 1/\sqrt{2}$ to circular polarization. In application to laser beams the profile function $a(u)$ is usually assumed to be smooth and slowly varying (on the scale of oscillations) and vanishing at $u \to \pm\infty$. In this article we are not intending to discuss a realistic laser and therefore, for the remainder of calculation, the pulse function is chosen to be constant $a(u) := a$. Formally this corresponds to a laser with an infinite length pulse.

Any solution to the classical equation of motion for a particle depends on the laser field's characteristics as well as on the initial/boundary conditions on a particle's position and velocity. The laser field's characteristics used below are, the frequency ω_L, the polarization ε and the gauge invariant dimensionless intensity parameter (3) which for our choice of monochromatic wave potential is

$$\eta^2 = -2\,\frac{e^2}{m^2c^4}\,\langle A_\mu\, A^\mu\rangle = \left(\frac{ea}{mc^2}\right)^2\,, \tag{51}$$

where now $\langle \dots \rangle$ denotes the time averaging over the period $2\pi/\omega_L$.

The dependence of a particle's orbits on the initial/boundary conditions is encoded via the first integrals $\mathbf{\Pi}_\perp$ and $\Pi_\|$. The analysis of a generic boundary conditions is given in the Appendix B. It is shown there how one can invert equation (40) expressing the auxiliary evolution parameter s in terms of the physical time t via the Weierstrass elliptic function $\wp(\omega_L t)$. Having this representation the remaining integrals (41) and (42) with the arbitrary constants $\mathbf{\Pi}_\perp$ and $\Pi_\|$ determine the explicit form of a particle's trajectory as function of the physical time t . However, to make the solution more transparent, we omit this rather tech-

nical work here and prefer to describe the orbits for a restricted but nevertheless informative initial conditions.

Recall that the solution (40)-(42) is written when the initial conditions on the coordinates (in physical time t) read

$$\boldsymbol{x}(t=0)=0\,, \tag{52}$$

and the initial velocity $\boldsymbol{v}(0):=\mathrm{d}\boldsymbol{x}/\mathrm{d}t(t=0)$ is expressible via the dimensionless constants of motion

$$\beta_+:=1-\frac{\Pi_\parallel}{mc}\,,\qquad \boldsymbol{\beta}_\perp=(\beta_1,\beta_2):=\frac{\boldsymbol{\Pi}_\perp}{mc}\,, \tag{53}$$

with the aid of relations

$$\boldsymbol{v}_\perp(0)\;=\;c\boldsymbol{\beta}_\perp-c\eta\,\boldsymbol{\epsilon}_\perp\,, \tag{54}$$

$$\upsilon_z(0)\;=\;c-c\sqrt{\beta_+^2-\eta^2\varepsilon^2+2\,\eta\,\boldsymbol{\epsilon}_\perp\cdot\boldsymbol{\beta}_\perp}\,, \tag{55}$$

where for the choice (50) we have $\boldsymbol{\epsilon}_\perp=(\varepsilon,0)$. Since for small velocities the system possesses a Galilean symmetry, (see Appendix A) we can pass to a certain reference frame by specifying the constants $\boldsymbol{v}(0)$. Below we restrict ourselves by insisting the vanishing of the transverse velocity[6]

$$\boldsymbol{\beta}_\perp=0\,. \tag{56}$$

Therefore, the orbits are specified by the constant,

$$\beta_z=\frac{\upsilon_z(0)}{c}=1-\sqrt{\beta_+^2-\eta^2\varepsilon^2}\,, \tag{57}$$

characterizing the particle's longitudinal velocity at $t=0$. In order to have a real velocity we require $\beta_+^2>\eta^2\varepsilon^2$.

Now with this specific choice of constants, we shall find $x(t)$, $y(t)$ and $z(t)$ as functions of the physical time t. Here we also note that due to the 2π- periodicity of the monochromatic gauge potential (50) all canonical coordinates will be treated as functions of a point on a circle with the standard trigonometric parametrization restricted to the principle domain $[-\pi/2\,,\pi/2]$.

[6] As it will be shown below the fixation $\boldsymbol{\beta}_\perp=0$, corresponds to zero average transverse velocity, $\langle\boldsymbol{v}_\perp\rangle=0$.

A. Fundamental domain and fundamental solution

Now we want to describe a particle's trajectory as a function of the laboratory frame's time t. First of all let us preserve the condition (27) which guaranties the monotonic character of the function $t(s)$ (see equation (12)), as well as reality of particle's trajectories is satisfied. For the monochromatic plane wave (50) the inequality (27) with the vanishing transverse momentum, $\mathbf{\Pi}_\perp = 0$, can be rewritten as

$$1 - \mu^2 \sin^2 u > 0\,, \tag{58}$$

where

$$\mu^2 := (1 - 2\,\varepsilon^2)\,\frac{\eta^2}{(1-\beta_z)^2}\,. \tag{59}$$

We define three allowed domains as

$$\text{(I)} \quad 0 < \mu^2 < 1\,, \quad \text{(II)} \qquad \mu^2 > 1\,, \qquad \text{(III)} \quad \mu^2 < 0\,.$$

Below we show that having the knowledge of the solution for region (I), which we call the *"fundamental domain"*, determines the solutions in all other regions. Namely, it will be demonstrated that all possible particle trajectories can be obtained from the *"fundamental solutions"* (solution depending on parameters from the "fundamental domain") by combination of inversion $\mu \to 1/\mu$, and rotation to the imaginary axis $\mu \to \imath\mu$.

Besides this, we also present solutions for two special cases

$$\mu^2 = 0 \qquad \text{and} \qquad \mu^2 = 1\,.$$

We emphasize here that these solutions can also be described by starting from the orbits from the fundamental domain and taking a corresponding limit.

To prove the above statements we start with the analysis of the "fundamental solutions" and derive a particle's trajectory in terms of the physical time t.

1. Orbits for the fundamental domain, (I) : $0 < \mu^2 < 1$.

For the parameters from the "fundamental domain" (I) the inequality (58) is true for all values of s from the interval $-\pi/2 \leq u \leq \pi/2$. Equations (40), (42) can be rewritten as

$$t(s) = \frac{1}{\omega_L(1-\beta_z)} \int_0^{\omega_L s} \mathrm{d}u \, \frac{1}{\sqrt{1-\mu^2\,\sin^2 u}} \,, \tag{60}$$

$$x(s) = -\frac{c}{\omega_L}\sqrt{\frac{\varepsilon^2}{1-2\varepsilon^2}}\ \arcsin\left[\sqrt{\mu^2}\,\sin(\omega_L s)\right] , \tag{61}$$

$$y(s) = \frac{c}{\omega_L}\sqrt{\frac{1-\varepsilon^2}{1-2\,\varepsilon^2}}\ \ln\left[\frac{\sqrt{\mu^2}\,\cos(\omega_L s)+\sqrt{1-\mu^2\sin^2(\omega_L s)}}{1+\sqrt{\mu^2}}\right] , \tag{62}$$

and the component in the wave propagation direction is

$$z(s) = ct(s) - cs \,. \tag{63}$$

This is the parametric solution to the equation of motion with parameter s from the principal interval

$$-\frac{\pi}{2} \leq \omega_L s \leq \frac{\pi}{2} \,. \tag{64}$$

Now, thanks to L. Euler, A.M. Legendre, N.H. Abel and C.G.J. Jacobi, we know how to invert (60), i.e. find the evolution parameter s as function of the physical time t . This can be done in terms of the well-known Jacobian *amplitude* function ([29, 30] and Appendix C)

$$\omega_L s = \mathrm{am}\big(\omega'_L t,\, \mu\big) \,, \tag{65}$$

with *modulus* $\mu := \sqrt{\mu^2}$ and the non-relativistically Doppler shifted frequency

$$\omega'_L := \omega_L\,(1-\beta_z) \,. \tag{66}$$

For the values of s from the interval (64) the amplitude function is a well defined increasing function defined on the interval

$$-\ \mathrm{K}(\mu)\ \leq \omega'_L t\ \leq \mathrm{K}(\mu) \,, \tag{67}$$

where K is the "real" quarter period of the Jacobian elliptic functions, (C7). Therefore, one can consider the transformation from the evolution parameter s to time t as well-defined change of coordinates on a circle.

Substituting the expression for the evolution parameter in terms of the physical time (65) into (61), (62) and (63) and using the properties of the Jacobian functions we arrive at the representation of the classical trajectory

$$x_{\mathrm{F}}(t) \;=\; -\frac{c}{\omega_L}\sqrt{\frac{\varepsilon^2}{1-2\,\varepsilon^2}}\;\arcsin\left[\mu\,\mathrm{sn}\big(\omega'_L t,\,\mu\big)\right]\,, \tag{68}$$

$$y_{\mathrm{F}}(t) \;=\; \frac{c}{\omega_L}\sqrt{\frac{1-\varepsilon^2}{1-2\,\varepsilon^2}}\;\ln\left[\frac{\mu\,\mathrm{cn}\big(\omega'_L t,\,\mu\big)+\mathrm{dn}\big(\omega'_L t,\,\mu\big)}{1+\mu}\right]\,, \tag{69}$$

and

$$z_{\mathrm{F}}(t) = ct - \frac{c}{\omega_L}\,\mathrm{am}\big(\omega'_L t,\,\mu\big)\,. \tag{70}$$

Here the subscript "F" is written to emphasize that (68)-(70) corresponds to the trajectories with the modulus from the fundamental interval $0 < \mu^2 < 1$.

Now we shall consider all the other domains of parameters and, using the properties of the Jacobian functions, will give explicit expressions for their corresponding trajectories.

2. Orbits for the second domain, (II) : $\mu^2 > 1$.

When analyzing this domain two peculiarities should be taken into account. First of all, when $\mu^2 > 1$ the inequality (58) is true only if

$$\sin^2 u < \underline{\mu}^2\,, \qquad \underline{\mu} := \frac{1}{\mu}\,. \tag{71}$$

This means that the upper limit to the integral in (60) lies in the interval

$$-\,\frac{\pi}{2} < -\arcsin\big(\underline{\mu}\big) \le\; \omega_L s\; \le \arcsin\big(\underline{\mu}\big) < \frac{\pi}{2}\,. \tag{72}$$

Second, since the standard integral representation of the amplitude function (C1) is defined for a modulus from the fundamental interval, $0 < \mu^2 < 1$, some simple mathematical manipulations are required to rewrite the integral (60) in such a form that its integrand depends on the inverse modulus, $\underline{\mu}$, instead of modulus μ . Namely one can easily verify that (40) can be written as

$$t(s) \;=\; \frac{1}{\omega'_L\mu}\int_0^{\arcsin(\mu\sin(\omega_L s))}\mathrm{d}u\,\frac{1}{\sqrt{1-\underline{\mu}^2\,\sin^2 u}}\,. \tag{73}$$

Therefore, the relationship between the evolution parameter s and time t is

$$\omega_L s = \arcsin\Big(\underline{\mu}\,\mathrm{sn}\,\big(\omega'_L\mu\, t,\,\underline{\mu}\big)\Big)\,. \tag{74}$$

When the parameter s is contained in the interval (72) relation (74) defines an increasing function on the interval

$$- \mathrm{K}(\underline{\mu}) \leq \omega'_L \mu \, t \leq \mathrm{K}(\underline{\mu}) \,. \tag{75}$$

Finally using (74) and relations (C2) we have

$$x(t) \;=\; -\frac{c}{\omega}\sqrt{\frac{\varepsilon^2}{1-2\,\varepsilon^2}}\,\mathrm{am}\big(\omega'_L \mu\, t,\, \underline{\mu}\big)\,, \tag{76}$$

$$y(t) \;=\; \frac{c}{\omega}\sqrt{\frac{1-\varepsilon^2}{1-2\,\varepsilon^2}}\;\ln\left[\frac{\mathrm{cn}\big(\omega'_L \mu\, t,\, \underline{\mu}\big) \;+ \mu\,\mathrm{dn}\big(\omega'_L \mu\, t,\, \underline{\mu}\big)}{\mu+1}\right]\,, \tag{77}$$

and

$$z(t) = ct - \frac{c}{\omega_L}\;\mathrm{arcsin}\,\Big(\underline{\mu}\,\mathrm{sn}\big(\omega'_L \mu\, t,\, \underline{\mu}\big)\Big)\;. \tag{78}$$

These formulae describe a particle's trajectory when the parameter μ^2 takes its value in the second domain, $\mu^2 > 1\,.$

3. *Orbits for the third domain,* (III) : $\mu^2 < 0\,.$

If $\mu^2 < 0$ the inequality (58) is true for the whole principal interval $[-\pi/2\,,\pi/2]$. Now introducing the positive definite parameter $\kappa^2 > 0\,, \mu^2 := -\kappa^2$, the relation (60) reads

$$t(s) \;=\; \frac{1}{\omega'_L}\int_0^{ws}\mathrm{d}u\,\frac{1}{\sqrt{1+\kappa^2\,\sin^2 u}}\,, \tag{79}$$

while the solution for the spatial coordinates is

$$x(s) \;=\; -\frac{c}{\omega'_L}\,\frac{\eta\varepsilon}{\kappa}\,\mathrm{arsinh}\left[\kappa\,\sin(\omega_L s)\right]\,, \tag{80}$$

$$y(s) \;=\; \frac{c}{\omega'_L}\,\frac{\sqrt{1-\varepsilon^2}}{i\kappa}\;\ln\left[\frac{\kappa\cos(\omega_L s)+i\sqrt{1+\kappa^2\sin^2(\omega_L s)}}{i+\kappa}\right]\,. \tag{81}$$

Now again a change of integration variable must be done in (79) in order to get the integral expressible in terms of the Jacobian amplitude with modulus from the open interval $(0, 1)$, thereby guaranteing that all functions are single-valued and continuous. It is straightforward to check that (79) is equivalent to

$$t(s) = \frac{1}{\omega'_L\kappa'}\int_0^{\phi(s)}\mathrm{d}u\,\frac{1}{\sqrt{1-\dfrac{\kappa^2}{\kappa'^2}\,\sin^2 u}}\,, \tag{82}$$

where the upper limit of the integral is

$$\phi(s) := \arcsin\left(\frac{\kappa' \sin(\omega s)}{\sqrt{1+\kappa^2\sin^2(\omega s)}}\right)\,, \qquad {\kappa'}^2 = 1+\kappa^2\,. \tag{83}$$

We have now achieved the goal that the modulus of $\kappa/\kappa' \in (0,1)$, for all values $\mu^2 < 0$. Therefore, the inverse to (82) reads

$$\phi(s) = \mathrm{am}\left(\omega'_L\kappa' t,\ \kappa/\kappa'\right)\,, \tag{84}$$

or directly for the evolution parameter we get

$$\kappa'\ \sin(\omega_L s) = \frac{\mathrm{sn}\left(\omega'_L\kappa' t,\ \kappa/\kappa'\right)}{\mathrm{dn}\left(\omega'_L\kappa' t,\ \kappa/\kappa'\right)}\,. \tag{85}$$

With the aid of (85) one can finally rewrite the particle's trajectory in terms of the Jacobian elliptic function with modulus κ/κ'

$$x(t) \;=\; -\frac{c}{\omega'_L}\,\frac{\eta\varepsilon}{\kappa}\mathrm{arsinh}\left[\frac{\kappa}{\kappa'}\,\frac{\mathrm{sn}\left(\omega'_L\kappa' t,\ \kappa/\kappa'\right)}{\mathrm{dn}\left(\omega'_L\kappa' t,\ \kappa/\kappa'\right)}\right]\,, \tag{86}$$

$$y(t) \;=\; \frac{c}{\omega_L}\,\frac{\eta\sqrt{1-\varepsilon^2}}{i\kappa}\,\ln\left[\frac{1-i\kappa\,\mathrm{cn}\left(\omega'_L\kappa'\, t,\ \kappa/\kappa'\right)}{(1-i\kappa)\mathrm{dn}\left(\omega'_L\kappa' t,\ \kappa/\kappa'\right)}\right]\,, \tag{87}$$

$$z(t) \;=\; ct - \frac{c}{\omega_L}\,\arcsin\left[\frac{1}{\kappa'}\,\frac{\mathrm{sn}\left(\omega'_L\kappa' t,\ \kappa/\kappa'\right)}{\mathrm{dn}\left(\omega'_L\kappa' t,\ \kappa/\kappa'\right)}\right]\,. \tag{88}$$

Equations (86)-(88) describe a particle's trajectory in the background characterized by $\mu^2 < 0$, and are well defined on the interval

$$-\,\mathrm{K}(\kappa/\kappa') < \omega'_L\kappa' t < \mathrm{K}(\kappa/\kappa')\,. \tag{89}$$

4. Degenerate orbits, $\mu^2 = 0$ & $\mu^2 = 1$.

Here we consider the two special cases remaining from the previous considerations. Namely, the special orbits with the parameter $\mu^2 = 0$ and $\mu^2 = 1$ will be presented. Note that the vanishing modulus corresponds to a special circularly polarized monochromatic plane wave, $\varepsilon^2 = 1/2$ or the trivial zero background case.

If $\mu^2 = 0$, then the parametric solution (40) determining the time t as a function of an auxiliary parameter s takes the simple form

$$t(s) = \frac{1}{1-\beta_z}\,s\,, \tag{90}$$

while the equations (42) for the spatial components orthogonal to the direction of the wave propagation reduce to

$$x(s) = -\frac{1}{\sqrt{2}}\frac{c}{\omega'_L}\,\eta\sin\omega_L s\,, \qquad y(s) = -\sqrt{2}\,\frac{c}{\omega'_L}\,\eta\,\sin^2\left(\frac{\omega_L s}{2}\right)\,. \tag{91}$$

Therefore, it follows from (90), that for the degenerate case of zero modulus $\mu^2 = 0$ the trajectory is

$$x(t) = -\frac{1}{\sqrt{2}}\,\eta\,\frac{c}{\omega'_L}\,\sin\omega'_L t\,, \quad y(t) = -\frac{2}{\sqrt{2}}\,\eta\frac{c}{\omega'_L}\sin^2\left(\frac{\omega'_L t}{2}\right), \quad z(t) = \beta_z\,c\,t.$$

From these expressions we see that for a circular polarized monochromatic wave all nonlinear effects disappear and one can choose such a reference frame where the particle's motion appears as a pure harmonic with the non-relativistically Doppler shifted laser frequency ω'_L.

For the orbit characterized by $\mu^2 = 1$, the equation (40) gives

$$t(s) = \frac{1}{\omega'_L}\,\mathrm{artanh}\left(\sin\omega_L s\right)\,, \tag{92}$$

and from (42) it follows that

$$x(s) = -\sqrt{\frac{\varepsilon^2}{1-2\,\varepsilon^2}}\,cs\,, \qquad y(s) = \sqrt{\frac{1-\varepsilon^2}{1-2\,\varepsilon^2}}\,\frac{c}{\omega_L}\ln\left(\cos\omega_L s\right)\,. \tag{93}$$

This finalizes our consideration of all possible particle's trajectories in a generic monochromatic plane wave.

In the next subsection we will briefly outline how all possible particle's trajectories can be categorized into the equivalent classes with respect to the action of the $SL(2,\mathbf{Z})/(1,-1)$ group.

B. Modular properties of orbits

As shown above, the nonlinear dependence on the polarization and the intensity of the radiation background of a particle's trajectory is encoded in the modulus of the elliptic Jacobian functions. The doubly periodic elliptic functions have a remarkable property related to a certain symmetry of modulus transformations. Namely, the elliptic function with periods w_1 and w_2 can be algebraically expressed through another elliptic functions with periods w'_1 and w'_2 if periods are related by the so-called unimodular transformations

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\,, \tag{94}$$

where the entries of the 2×2 matrix are integers $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The transformations (94) may also be treated also as a subgroup of the Möbius transformations in the upper half of the complex τ plane into itself

$$\tau' = \frac{c + d\tau}{a + b\tau}\,, \qquad \tau := \frac{w_2}{w_1}\,. \tag{95}$$

For the problem we are dealing with here this modular equivalence exposes in the *intensity duality*, i.e. there exist a specific correspondence between the motion in backgrounds with various intensity regimes. Any trajectory with an arbitrarily prescribed intensity can be connected to the solution from the fundamental domain with the aid of a certain modular transformation. Particularly, if we assign to a particle's trajectory with low intensity a certain modular parameter τ, then the trajectory for the high intensity conditions are related by $\tau \to \tau/(1+\tau)$. Indeed, using the relations between the Jacobian functions whose moduli are inverse to each other, (cf. formulae (C27)), one can verify that the expressions (76)-(78) with $\mu^2 > 1$ follow from the fundamental solution (68)-(70) by the substitution $\mu \to 1/\mu$:

$$\boldsymbol{x}(t\,|\,\mu) = \boldsymbol{x}_{\mathrm{F}}(t\,|\,\frac{1}{\mu}\,)\,, \tag{96}$$

or in terms of the modular parameter τ

$$\boldsymbol{x}(t\,|\,\tau) = \boldsymbol{x}_{\mathrm{F}}(t\,|\,\frac{\tau}{1+\tau}\,)\,. \tag{97}$$

Analogously, if $\mu^2 < 0$, the trajectories (86)-(88) are connected to the fundamental solution by the shift transformation (C26),

$$\boldsymbol{x}(t\,|\,\tau) = \boldsymbol{x}_{\mathrm{F}}(t\,|1+\tau\,)\,. \tag{98}$$

Note also that the special trajectories with $\mu^2 = 0$ and $\mu^2 = 1$, considered above, coincides with the corresponding limits of the fundamental solution taking into account that the Jacobian functions are degenerate to the trigonometric (C13) and hyperbolic functions (C12) for moduli $\mu = 0$ and $\mu = 1$, respectively.

IV. ANALYSIS OF THE PARTICLE'S TRAJECTORY

Now we analyze in greater detail the trajectory with parameters from the fundamental domain and clarify some of the physical features of this solution.

From the solution (68)-(70) and (C4)-(C6) one can derive an expression for the particle's velocity

$$
\begin{aligned}
v_x(t) &= -c\eta\epsilon \,\mathrm{cn}\left(\omega'_L t,\, \mu\right) , && (99)\\
v_y(t) &= -c\eta\sqrt{1-\varepsilon^2}\,\mathrm{sn}\left(\omega'_L t,\, \mu\right) , && (100)\\
v_z(t) &= c - c(1-\beta_z)\,\mathrm{dn}\left(\omega'_L t,\, \mu\right) . && (101)
\end{aligned}
$$

Periodic properties of the Jacobian function (see eqs. (C9) in the Appendix C) tell us that the components of the charged particle's velocity in the plane orthogonal to the wave propagation are periodic functions of time with period

$$
T_P := \frac{4\mathrm{K}}{\omega'_L} = \frac{2\pi}{\omega_P} , \tag{102}
$$

while in the direction of propagation the oscillation's period is half the size $T_{\mathrm{P}}/2$. The fundamental circular frequency of the particle's motion, ω_{P}, differs from the frequency of the laser field

$$
\omega_P = \frac{\pi}{2\mathrm{K}}\,\omega'_L . \tag{103}
$$

From this expression we see that, apart from the pure kinematical non-relativistic Doppler shift (66), the frequency of a particle's oscillation depends on the laser's intensity through the real quarter period K (C7) of the Jacobian functions. The presence of K in (103) exposes a new property of a particle's dynamics which is beyond the dipole approximation. A particle oscillates at frequency that depends on the laser intensity and polarization in a nonlinear way. For the low intensity regime, $\eta \ll 1$, the period of a particle's oscillation can be represent in the form of an expansion (C18) as

$$
T_P = \frac{2\pi}{\omega'_L}\left[1 + \left(\frac{1}{2}\right)^2 \frac{1-2\,\epsilon^2}{(1-\beta_z)^2}\,\eta^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{(1-2\,\epsilon^2)^2}{(1-\beta_z)^4}\,\eta^4 + \dots\right] . \tag{104}
$$

It is well-known that, in contrast to the dipole approximation where the particle's motion is a pure harmonic, the relativistic dynamics exhibits a drift action of the laser field on the particle which depends on the intensity [12, 31, 32]. In our consideration, when the relativistic effects are partially taken into account, this effect also can be seen. For the non-relativistic case with our choice of the initial conditions, $\mathbf{\Pi}_\perp = 0$, the mean velocity for the transverse direction vanishes

$$
\langle \boldsymbol{v}_\perp \rangle = 0 . \tag{105}
$$

While the drift in the direction of propagation is a nonlinear function of the laser beam's intensity

$$\langle \upsilon_z \rangle = c - \frac{\pi c}{2\mathrm{K}}(1-\beta_z)\,. \tag{106}$$

This drift velocity $\langle \upsilon_z \rangle$ for small intensities at leading order reads

$$\langle \upsilon_z \rangle = \upsilon_z(0)\left(1 - \frac{1-2\,\epsilon^2}{4(1-\beta_z)^2}\,\eta^2\right) + \dots\,. \tag{107}$$

Another new feature when comparing to the the dipole-approximation is the appearance of the higher harmonics in the particle's motion. This can be seen from our solution with the aid of the well-known Fourier series expansion for the Jacobian function [29, 30]. Using these formulae collected in Appendix C one can represent the trajectory as

$$x(t) \;=\; \frac{4c\epsilon}{\omega_L\sqrt{1-2\,\epsilon^2}}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{(2n-1)(1+q^{2n-1})}\,\sin(2n-1)\omega_P\,t\,, \tag{108}$$

$$y(t) \;=\; \frac{8c\sqrt{1-\epsilon^2}}{\omega_L}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{(2n-1)(1-q^{2n-1})}\,\sin^2\left(n-\frac{1}{2}\right)\omega_P\,t\,, \tag{109}$$

$$z(t) \;=\; \langle \upsilon_z \rangle\, t - \frac{c}{\omega_L}\sum_{n=1}^{\infty}\frac{2q^n}{n(1+q^{2n})}\,\sin 2n\,\omega_P\,t\,, \tag{110}$$

where q is the so-called *nome* parameter

$$q := \exp\left(-\pi\,\frac{\mathrm{K}'}{\mathrm{K}}\right)\,. \tag{111}$$

Note (see eq. (C22)) that the nome q for small intensities is approximately

$$q \approx \frac{1-2\,\epsilon^2}{16(1-\beta_z)^2}\,\eta^2 + O(\eta^4)\,. \tag{112}$$

When the intensity parameter is small one can perform a Galilean boost with the velocity $\boldsymbol{V} := -\left(0,0,\langle \upsilon_z \rangle\right)$ to the so-called *average rest frame* (ARF), frame where the mean particle's velocity vanishes. In this frame the particle's motion represents only the superposition of harmonic oscillations with the fundamental frequency ω_P.

In the ARF frame equations (108)-(110) reduce, for small intensities, to the following expressions in the leading order of the η-expansion

$$x_{\text{ARF}}(t) \;=\; -\frac{c\varepsilon}{\omega_L'}\,\eta\sin\omega_L' t\,, \tag{113}$$

$$y_{\text{ARF}}(t) \;=-\; \frac{c\sqrt{1-\varepsilon^2}}{\omega_L'}\,\eta\,(1-\cos\omega_L' t)\,, \tag{114}$$

$$z_{\text{ARF}}(t) \;=\; \frac{c}{\omega_L'}\,\frac{1-2\,\epsilon^2}{8(1-\beta_z)}\,\eta^2\,\sin 2\omega_L' t\,. \tag{115}$$

Pictorially these formulae describe the orbits shaped like a figure of eight. This form of the orbit is well-known from the parametric relativistic solution [8, 9, 10] and (113) -(115) are the small intensity approximation to the explicit representation of the relativistic trajectories. We omit here the detailed comparison with the relativistic solution since it requires the knowledge of a relativistic particle's trajectory as a function of the physical time as well as a careful analysis of the frame dependence of our partially relativistic solution. We intend to do this in a forthcoming publication.

All mentioned features, the Doppler shift, dependence of the particle's oscillation frequency on the laser beam's intensity as well as the presence of higher harmonics in the particle's motion, lead to a several important phenomena. Among them there are the nonlinear modification to the classical Thomson scattering and a charged particle's mass/energy shift in the electromagnetic background radiation.

V. CONCLUDING REMARKS

Most textbooks on the classical dynamics of particles pay tribute to a historical tradition. They begin with the consideration of non-relativistic mechanics and, after introducing the basics of relativity, discuss the specific features of the corresponding relativistic problem. The exception to this rule is the charged particle's dynamics in the background of an electromagnetic plane wave. All well-known sources either discuss the problem in the dipole approximation [4] or solve the problem in the framework of relativistic mechanics directly [8, 9, 10].

In the present note we "restore historical justice" by considering the classical problem of a Newtonian particle's motion in a given electromagnetic plane wave field beyond the above mentioned dipole approximation. We obtain an exact representation for a particle's orbit in the parametric form which is analogous to the well-known relativistic solution. Furthermore, we show that the three-dimensional non-relativistic orbit of a particle's motion in a monochromatic arbitrarily polarized plane wave admits, as an explicit function of the laboratory frame's time, the exact solution in terms of the doubly periodic elliptic functions. This is in contrast to the relativistic trajectory which is known explicitly only in a four dimensional parametric form. The derived solution explicitly exposes the presence of higher fundamental harmonics in a charged particle's motion as well as nonlinear dependence of the

oscillations on the intensity and polarization of the monochromatic background. Another interesting characteristic of the trajectory we discovered is an intensity duality between motion in different radiation backgrounds. Since a given intensity defines the modulus of the Jacobian functions, a particle's trajectory in backgrounds whose intensities are connected by the modular transformations are simply related.

Finally it is worth commenting on the sinuous way to derive the exact particle trajectories within the reparametrization invariant form of the Hamilton-Jacobi method presented here. We prefer to follow it since this approach incorporates both a classical as well as a quantum treatment. The knowledge of the classical generating function connecting the interacting and free systems helps to construct the corresponding quasiclassical description and in particular can be exploited for study of the laser-atom interactions [1]. Furthermore, the suggested approach mimics the four dimensional relativistic treatment which is useful to study the deviation from non-relativistic motion. All these issues we intend to discuss in detail in future publications. Apart from this, based on the explicit form of the non-relativistic trajectory we plan to study several important effects, including the non-linear Thomson/Compton scattering, electromagnetic dressing, charged particle acceleration by a laser field etc., in the transition regimes between the non-relativistic and relativistic cases.

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APPENDIX A: REMARKS ON "NON-RELATIVISTIC" SYMMETRIES

Below we discuss the realization of the gauge symmetry as well as the implementation of the Galilean boost for a charged "non-relativistic" particle travelling in an electromagnetic background.

1. Gauge transformations

In terms of the 3-dimensional notations for a gauge potential $A^\mu = (\Phi\,,\,\boldsymbol{A})$ and coordinates $x^\mu = (ct\,, x, y, z) = (ct\,, \boldsymbol{x})$, the gauge transformation reads

$$\boldsymbol{A}(t,\boldsymbol{x}) \to \boldsymbol{A}'(t,\boldsymbol{x}) \;=\; \boldsymbol{A}(t,\boldsymbol{x}) + \frac{\partial}{\partial \boldsymbol{x}}\,\Omega(t,\boldsymbol{x})\,, \tag{A1}$$

$$\Phi(t,\boldsymbol{x}) \to \Phi'(t,\boldsymbol{x}) \;=\; \Phi(t,\boldsymbol{x}) - \frac{1}{c}\frac{\partial}{\partial t}\,\Omega(t,\boldsymbol{x})\,. \tag{A2}$$

Under the transformations (A1) and (A2) the "non-relativistic" Lagrangian (5) *is not invariant*:

$$\mathcal{L}\left(\boldsymbol{x}\,, \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\,, t\right) \to \mathcal{L}\left(\boldsymbol{x}\,, \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\,, t\right) + \frac{e}{c}\,\frac{\mathrm{d}}{\mathrm{d}t}\Omega(t\,,\boldsymbol{x}(t))\,. \tag{A3}$$

Here $\mathrm{d}/\mathrm{d}t$ denotes the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} := \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \frac{\partial}{\partial t}\,. \tag{A4}$$

However, the variation (A3) being the total derivative of the function $e/c\,\Omega(t\,,\boldsymbol{x}(t))$, does not affect to the classical equations of motion.

The changes of gauge potentials (A1) and (A2) are canonical transformations with the generating function

$$F(\boldsymbol{x},\boldsymbol{P},t) = \boldsymbol{x}\cdot\boldsymbol{P} - \Omega(t\,,\boldsymbol{x}(t))\,. \tag{A5}$$

2. Galilean boosts

Under a Lorentz boost in the x direction with the factor $\gamma := 1/\sqrt{1 - V^2/c^2}$

$$t' = \gamma\left(t - \frac{V}{c^2}\,x\right)\,, \qquad x' = \gamma\,(x - Vt)\,, \qquad y' = y\,, \qquad z' = z\,, \tag{A6}$$

a gauge potential $A^\mu = (\Phi, \boldsymbol{A})$ being a Lorentz 4-vector transforms as

$$\Phi' = \gamma\left(\Phi - \frac{V}{c}\,A_x\right)\,, \qquad A_x{}' = \gamma\left(A_x - \frac{V}{c}\,\Phi\right)\,, \qquad A_y{}' = A_y\,, \qquad A_z{}' = A_z\,. \tag{A7}$$

For small velocities, neglecting terms of order V^2/c^2 and higher the Lorentz boost (A6) reduces to the Galilean boost

$$t' = t\,, \qquad x' = x - Vt\,, \qquad y' = y\,, \quad z' = z\,, \tag{A8}$$

while the gauge potential changes as

$$\Phi' = \Phi - \frac{V}{c} A_x\,, \qquad {A_x}' = A_x - \frac{V}{c}\,\Phi\,, \qquad {A_y}' = A_y\,, \qquad {A_z}' = A_z\,. \tag{A9}$$

With the help of these relations, keeping only the leading terms in V/c, we have

$$\mathcal{L} \to \mathcal{L}' \;=\; \mathcal{L} - m\frac{\mathrm{d}}{\mathrm{d}t}\left(V\,x - \frac{1}{2}\,V^2 t\right)\,, \tag{A10}$$

note that V is *not* infinitesimally small. The Lagrangian is invariant under the Galilean boost *up to a total derivative* only.

Both variations of the Lagrangian, under the gauge transformations (A3) as well as under the Galilean boost (A10), are not important at the classical level but can lead in general to a nontrivial quantum phenomena. Particularly, they lead to the appearance of the so-called 1-cocycle for the wave function. The action of the unitary operator U_G generating the Galilean boost on the wave function reads

$$U_G\Psi(\boldsymbol{x}) = \exp i\left(m\boldsymbol{V}\cdot\boldsymbol{x} - \frac{1}{2}m\boldsymbol{V}^2 t\right)\Psi(\boldsymbol{x} - \boldsymbol{V}t)\,. \tag{A11}$$

It is interesting that for the charged particle this cocycle can be trivialized. Indeed, if the Galilean boost is accompanied by the gauge transformation generated by (A5) with the special gauge function

$$\Omega(t, \boldsymbol{x}(t)) := \frac{e}{c}\left(m\boldsymbol{V}\cdot\boldsymbol{x} - \frac{1}{2}m\boldsymbol{V}^2 t\right)\,,$$

then the Lagrangian (5) remains unchanged and cocycle is removed.

APPENDIX B: GENERIC BOUNDARY CONDITIONS

Here we briefly state how one can express the physical time t in terms of the auxiliary evolution parameter when the generic boundary conditions on a particle's position and velocity are imposed. To find this dependance one can proceed as follows. Plugging the expression for the gauge potential (50) into the function $W(u,\Pi_\perp)$ from (26) we have

$$W(u,\Pi_\perp) = -\eta^2 m^2 c^2\left(\epsilon^2\cos^2 u + (1-\epsilon^2)\sin^2 u\right) + 2\eta mc\left(\Pi_1\cos u + \Pi_2 \sin u\right)\,.$$

The integral in (40) with this expression represents the so-called elliptical integral. Indeed, using the universal trigonometric substitution $x = \tan(u/2)$ it can be rewritten in the

Weierstrass form:

$$t(s) = \frac{2}{\omega_L} \int_0^{\tan(\omega_L s/2)} \mathrm{d}x \, \frac{1}{\sqrt{f(x)}} \,, \tag{B1}$$

with the fourth order polynomial

$$f(x) = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 \,, \tag{B2}$$

whose coefficients are

$$a_0 := \beta_+^2 - \eta^2 \varepsilon^2 - 2\,\eta\,\beta_1 \,, \qquad a_1 := \eta\,\beta_2 \,, \qquad a_3 := \eta\,\beta_2 \,, \tag{B3}$$

$$a_2 := \frac{1}{3}\beta_+^2 + \eta^2 \left(\varepsilon^2 - \frac{2}{3}\right) , \qquad a_4 := \beta_+^2 - \eta^2 \varepsilon^2 + 2\,\eta\,\beta_1 \,. \tag{B4}$$

According to the classical result (see e.g. [29]) attributed to Weierstrass, the integral (B1) can be inverted:

$$\tan\left(\frac{\omega_L s}{2}\right) = \frac{\sqrt{f(0)}\,\wp'\left(\frac{\omega_L}{2}\,t\right) - \frac{1}{2}\left[\wp\left(\frac{\omega_L}{2}\,t\right) - \frac{1}{24} f''(0)\right] + \frac{1}{24} f(0) f'''(0)}{2\left[\wp\left(\frac{\omega_L}{2}\,t\right) - \frac{1}{24} f''(0)\right]^2 - \frac{1}{48} f(0) f''''(0)} \,. \tag{B5}$$

Here the number of primes over the polynomial (B2) denotes the order of the derivatives with respect to x. In (B5) the Weierstrass doubly periodic function $\wp\,(z\,; g_2\,, g_3)$ depends on two invariants g_2 and g_3 of the polynomial (B2)

$$\begin{aligned}
g_2 :\; &= 4\,\eta^4 \left(\varepsilon^4 - \varepsilon^2 + \frac{1}{3}\right) - 4\eta^2 \left(\frac{1}{3}\beta_+^2 + \boldsymbol{\beta}_\perp^2\right) + \frac{4}{3}\beta_+^4 \,, \\
g_3 :\; &= \frac{4}{3}\eta^6 \left(\varepsilon^4 - \varepsilon^2 + \frac{2}{9}\right) - 4\eta^4 \left[\frac{2}{3}\,\varepsilon^4 \beta_+^2 + \varepsilon^2 \left(\beta_1^2 - \beta_2^2 - \frac{2}{3}\beta_+^2\right) + \frac{1}{9}\beta_+^2 - \frac{2}{3}\,\beta_1^2 + \frac{1}{3}\,\beta_2^2\right] \\
&- \frac{4}{3}\,\eta^2 \beta_+^2 \left(\frac{1}{3}\beta_+^2 + \boldsymbol{\beta}_\perp^2\right) + \frac{8}{27}\,\beta_+^6 \,.
\end{aligned}$$

These equations show that the dependence of any particle's trajectory on the initial conditions is accumulated in the invariants g_2, g_3 and functionally is quite subtle. Particulary there is no rotational degeneracy with respect to the vector $\mathbf{\Pi}_\perp$.

APPENDIX C: VOCABULARY ON THE JACOBIAN ELLIPTIC FUNCTIONS

Following the classical textbooks [29, 30] and handbook [33] we collect the basic formulae on Jacobian elliptic functions which are extensively used in the main text.

The *amplitude function* $\phi := \mathrm{am}(z\,,\mu)$ is the *inverse* of the function defined by the integral

$$z(\phi) = \int_0^{\phi} \mathrm{d}\vartheta \frac{1}{\sqrt{1-\mu^2\sin^2\vartheta}}\,. \tag{C1}$$

The amplitude $\mathrm{am}(z\,,\mu)$ is an infinitely many-valued function whose principal domain of definition for real z is $(-\mathrm{K}\,,\mathrm{K})$. The value of this constant K is determined by the *modulus* μ via the so-called *complete elliptic integral*, see (C7) below.

Three basic *Jacobian elliptic functions*, $\mathrm{sn}(z,\mu)$, $\mathrm{cn}(z,\mu)$ and $\mathrm{dn}(z,\mu)$ are analytical functions of the complex variable z everywhere except at the simple poles. These elliptic function are expressible in terms of the amplitude function

$$\mathrm{sn}(z,\mu) = \sin\big(\mathrm{am}(z\,,\mu)\big)\,, \quad \mathrm{cn}(z,\mu) = \cos\big(\mathrm{am}(z\,,\mu)\big)\,, \quad \frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{am}(z,\mu) = \mathrm{dn}(z,\mu)\,, \tag{C2}$$

and satisfy the basic algebraic

$$\mathrm{sn}^2 z + \mathrm{cn}^2 z = 1\,, \quad \mu^2\mathrm{sn}^2 z + \mathrm{dn}^2 z = 1\,, \tag{C3}$$

and differential relations

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{sn}(z,\mu) \;=\; \mathrm{cn}(z\,,\mu)\,\mathrm{dn}(z\,,\mu)\,, \tag{C4}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{cn}(z,\mu) \;=\; -\mathrm{sn}(z\,,\mu)\,\mathrm{dn}(z\,,\mu)\,, \tag{C5}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{dn}(z,\mu) \;=\; -\mu^2\mathrm{sn}(z\,,\mu)\,\mathrm{cn}(z\,,\mu)\,, \tag{C6}$$

which show the analogy of the Jacobian functions to the trigonometric functions.

Functions $\mathrm{sn}(z,\mu)$, $\mathrm{cn}(z,\mu)$ and $\mathrm{dn}(z,\mu)$ are doubly periodic functions of z. Periods of $\mathrm{sn}(z,\mu)$ are $4\mathrm{K}$ and $2i\mathrm{K}'$ while periods of $\mathrm{cn}(z,\mu)$ are $4\mathrm{K}$ and $2\mathrm{K}+2i\mathrm{K}'$. Function $\mathrm{dn}(z,\mu)$ has periods $2\mathrm{K}$ and $4i\mathrm{K}'$. The "real" (K) and "imaginary" (K') quarter periods are real numbers given by the complete elliptic integrals

$$\mathrm{K}(\mu) \;:=\; \int_0^{\pi/2} \mathrm{d}\vartheta \frac{1}{\sqrt{1-\mu^2\sin^2\vartheta}}\,, \tag{C7}$$

$$i\,\mathrm{K}'(\mu) \;:=\; i\int_0^{\pi/2} \mathrm{d}\vartheta \frac{1}{\sqrt{1-(1-\mu^2)\sin^2\vartheta}}\,. \tag{C8}$$

The Jacobian functions as functions of the modulus are single valued on the complex μ plane with two cuts $[1,\infty)$ and $(-\infty,0]$.

The discrete symmetry. The Jacobian functions cn and dn are even functions , while sn is odd. They obey the relations

$$\mathrm{sn}(u + 2m\mathrm{K} + 2ni\mathrm{K}' \,, \mu) \;=\; (-)^m \,\mathrm{sn}(u \,, \mu)\,, \tag{C9}$$

$$\mathrm{cn}(u + 2m\mathrm{K} + 2ni\mathrm{K}' \,, \mu) \;=\; (-)^{m+n} \,\mathrm{cn}(u \,, \mu)\,, \tag{C10}$$

$$\mathrm{dn}(u + 2m\mathrm{K} + 2ni\mathrm{K}' \,, \mu) \;=\; (-)^n \,\mathrm{dn}(u \,, \mu)\,, \tag{C11}$$

where $n, m \in \mathbb{Z}$.

Two degenerate cases. The doubly periodic Jacobian functions degenerate to other functions when one of the periods become infinite, that if μ is 0 or 1 .

When $\mu = 1$ the real quarter-period $\mathrm{K} = \infty$ and Jacobian function degenerate to the hyperbolic functions

$$\mathrm{sn}(u, 1) = \tanh u\,, \qquad \mathrm{cn}(u, 1) = \frac{1}{\cosh u}\,, \qquad \mathrm{dn}(u, 1) = \frac{1}{\cosh u}\,. \tag{C12}$$

If $\mu = 0$ the real quarter-period is finite, $\mathrm{K} = \pi/2$, but the imaginary quarter-period is infinite and the Jacobian function degenerate to the trigonometric functions

$$\mathrm{sn}(u, 0) = \sin u\,, \qquad \mathrm{cn}(u, 0) = \cos u\,, \qquad \mathrm{dn}(u, 0) = 1\,, \tag{C13}$$

Small modulus expansions. When modulus is small enough the Jacobian functions can be approximated as

$$\mathrm{am}(z, \mu) \;=\; z - \frac{1}{4}\,\mu^2\,[z - \sin(z)\cos(z)] + o(\mu^4)\,, \tag{C14}$$

$$\mathrm{sn}(z, \mu) \;=\; \sin(z) - \frac{1}{4}\,\mu^2\,[z - \sin(z)\cos(z)]\cos(z) + o(\mu^4)\,, \tag{C15}$$

$$\mathrm{cn}(z, \mu) \;=\; \cos(z) + \frac{1}{4}\,\mu^2\,[z - \sin(z)\cos(z)]\cos(z) + o(\mu^4)\,, \tag{C16}$$

$$\mathrm{dn}(z, \mu) \;=\; 1 - \frac{1}{2}\,\mu^2 \sin^2(z) + o(\mu^4)\,. \tag{C17}$$

The quarter period K has a small modulus expansion of the form

$$\mathrm{K} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 \mu^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \mu^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \mu^6 + \ldots\right]. \tag{C18}$$

The Fourier series expansions: These are

$$\mathrm{sn}(z,\,\mu) \;=\; \frac{2\pi}{\mu \mathrm{K}}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1-q^{2n-1}}\,\sin(2n-1)\frac{\pi z}{2\mathrm{K}}\,, \tag{C19}$$

$$\mathrm{cn}(z,\,\mu) \;=\; \frac{2\pi}{\mu \mathrm{K}}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1+q^{2n-1}}\,\cos(2n-1)\frac{\pi z}{2\mathrm{K}}\,, \tag{C20}$$

$$\mathrm{dn}(z,\,\mu) \;=\; \frac{2\pi}{\mathrm{K}}+\frac{2\pi}{\mathrm{K}}\sum_{n=1}^{\infty}\frac{q^{n}}{1+q^{2n}}\,\cos 2n\,\frac{\pi z}{2\mathrm{K}}\,, \tag{C21}$$

with the so-called *nome*, or Jacobi parameter $q=\exp\left(-\pi\mathrm{K}'/\mathrm{K}\right)$. Similarly to the quarter periods the nome q can be expanded in powers of the modulus

$$q=\frac{\mu^2}{16}+8\left(\frac{\mu^2}{16}\right)^2+84\left(\frac{\mu^2}{16}\right)^3+\dots\,, \tag{C22}$$

The modular transformations. The ratio $\tau := i\mathrm{K}'/\mathrm{K}$ serves as an important modular parameter[7]. Under the unimodular transformation

$$\tau\to\frac{c+d\tau}{a+b\tau}\,,\qquad a,b,c,d\in\mathbb{Z}\,,\qquad ad-bc=1\,, \tag{C23}$$

the doubly periodic elliptic functions with different periods are expressible through each other.

Any transformation (C23) can be represented as a product of powers of two generators

$$\mathrm{S}:=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\,,\qquad \mathrm{T}:=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\,. \tag{C24}$$

The matrix S, generates the so-called Jacobi imaginary transformation

$$\tau\to\tau':=-1/\tau\,,$$

under which the Jacobian functions vary as

$$\mathrm{sn}(iz,\mu')=i\frac{\mathrm{sn}(z,\mu)}{\mathrm{cn}(z,\mu)},\quad \mathrm{cn}(iz,\mu')=\frac{1}{\mathrm{cn}(z,\mu)},\quad \mathrm{dn}(iz,\mu')=\frac{\mathrm{dn}(z,\mu)}{\mathrm{cn}(z,\mu)}, \tag{C25}$$

where $\mu' := \sqrt{1-\mu^2}$, the *complementary modulus.* The action of the generator T which represents the shift transformation

$$\tau\to\tau':=1+\tau\,,$$

[7] We assume that $\mathrm{Re}(\tau)>0$.

results in the following relations between the basic Jacobian functions

$$\mathrm{sn}(\mu' z, \frac{i\mu}{\mu'}) = \mu' \frac{\mathrm{sn}(z, \mu)}{\mathrm{dn}(z, \mu)}, \quad \mathrm{cn}(\mu' z\,, \frac{i\mu}{\mu'}) = \frac{\mathrm{cn}(z, \mu)}{\mathrm{dn}(z, \mu)}, \quad \mathrm{dn}(\mu' z, \frac{i\mu}{\mu'}) = \frac{1}{\mathrm{dn}(z, \mu)}. \tag{C26}$$

Finally the change

$$\tau \to \tau' := \tau/(1+\tau)\,,$$

which can be represented as, TST, is of special interest to us since it relates the functions with the inverse moduli μ and $\underline{\mu} := \mu^{-1}$:

$$\mathrm{sn}(\mu z\,, \underline{\mu}) = \mu\, \mathrm{sn}(z\,, \mu)\,, \quad \mathrm{cn}(\mu z\,, \underline{\mu}) = \mathrm{dn}(z\,, \mu)\,, \quad \mathrm{dn}(\mu z\,, \underline{\mu}) = \mathrm{cn}(z\,, \mu)\,. \tag{C27}$$

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